Space–adiabatic theory for random–Landau Hamiltonian: results and prospects

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The study of the (integer) *Quantum Hall Effect* (QHE) requires a careful analysis of the spectral properties of the $2D$, single-electron Hamiltonian

$$H_{\Gamma,B} := \left(-\mathrm{i}\partial_x - B\ y\right)^2 + \left(-\mathrm{i}\partial_y + B\ x\right)^2 + V_\Gamma(x,y) \tag{0.1}$$

where $H_B := H_{\Gamma,B} - V_\Gamma$ is the usual *Landau Hamiltonian* (in symmetric gauge) with *magnetic field* B and V_Γ is a $\Gamma \equiv \mathbb{Z}^2$ *periodic potential* which models the electronic interaction with a crystalline structure. Under usual conditions (e.g., $V_\Gamma \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$) the Hamiltonian (0.1) is self-adjoint on a suitable subdomain of $L^2(\mathbb{R}^2)$. A direct analysis of the fine spectral properties of (0.1) is extremely difficult and one needs resorting to simpler effective models hoping to capture (some of) the main physical features in suitable physical regimes.

Weak magnetic field limit. The regime $B \ll 1$ is very interesting since it is easily accessible to experiments. The common lore, (cf. works of R. Peierls, P. G. Harper and D. Hofstadter), says that the "local description" of the spectrum of (0.1) is "well approximated" by the spectrum of the *Hofstadter* (effective) *model*

$$\left(H^{(B)}_{\mathrm{Hof}}\xi\right)_{n,m} := e^m_B\ \xi_{n+1,m} + \overline{e^m_B}\ \xi_{n-1,m} + \overline{e^n_B}\ \xi_{n,m+1} + e^n_B\ \xi_{n,m-1} \tag{0.2}$$

with $\{\xi_{n,m}\} \in \ell^2(\mathbb{Z}^2)$ and $e^m_B := \mathrm{e}^{\mathrm{i}2\pi mB}$.

The above discussion leads to the following questions:

- Q.1) In what mathematical sense are $\mathrm{Spec}\big(H_{\Gamma,B}\big)$ and $\mathrm{Spec}\big(H^{(B)}_{\mathrm{Hof}}\big)$ "locally equivalent"?
- Q.2) What is the relation between the "effective" dynamics induced by $H^{(B)}_{\mathrm{Hof}}$ and the "true" dynamics induced by $H_{\Gamma,B}$?

A third question concerns the rôle of the disorder in the explanation of the QHE. Indeed, the introduction of a *random potential* V_ω (e.g., an *Anderson potential*) in (0.1), leading to

$$H_{\Gamma,B,\omega} := H_{\Gamma,B} + V_\omega, \tag{0.3}$$

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is essential in order to explain the emergence of the quantum Hall plateaus. Then:

Q.3) In presence of disorder is it still possible to derive a "simplified" (i.e., effective) model for $H_{\Gamma,B,\omega}$ which encodes the (main) spectral and dynamical properties of the full model?

To answer to questions Q.1) and Q.2) one needs to prove the so-called *Peierls substitution*. This is an old problem which dates back to the works of J. Bellissard [**1**] and B. Helffer and J. Sjöstrand [**6**]. However, these works provide only a partial answer to Q.1) (*local isospectrality*) and no answer for Q.2). A complete solution has been given only recently by the author and M. Lein in [**3**]. In this paper a strong version of the Peierls substitution is derived by means of a joint application of the *Space-adiabatic perturbation theory* (SAPT) developed by G. Panati, H. Spohn and S. Teufel [**8**] and the *magnetic Weyl quantization* developed by M. Măntoiu and R. Purice [**7**]. The main result derived in [**3**] can be stated as follows:

Theorem 0.1. *Assume the existence of a $S \subset \mathrm{Spec}\big(H_{\Gamma,B=0}\big)$ separated from the rest of the spectrum $\mathrm{Spec}\big(H_{\Gamma,B=0}\big) \setminus S$ by gaps*[1]*. Then:*

(i) *Associated to S there exists an an orthogonal projection Π_B in $L^2(\mathbb{R}^2)$ such that for any $N \in \mathbb{N}$*

$$\big\|[H_{\Gamma,B};\Pi_B]\big\| \leqslant C_N\ B^N \qquad \textit{if} \qquad B \to 0 \tag{0.4}$$

where $C_N > 0$ are suitable constants. The space $\mathrm{Ran}\ \Pi_B \subset L^2(\mathbb{R}^2)$ is called almost-invariant *subspace.*

(ii) *There exists a* reference *Hilbert space $\mathcal{H}_{\mathrm{r}}$ (B-independent), an* effective *(bounded)* operator *H^{eff}_B on $\mathcal{H}_{\mathrm{r}}$ and a unitary operator $U_B : \mathrm{Ran}\ \Pi_B \to \mathcal{H}_{\mathrm{r}}$ such that for any $N \in \mathbb{N}$*

$$\big\|\big(\mathrm{e}^{\mathrm{i}tH_{\Gamma,B}} - U_B^{-1}\ \mathrm{e}^{\mathrm{i}tH^{\mathrm{eff}}_B}\ U_B\big)\Pi_B\big\| \leqslant C_N\ B^N\ |t| \qquad \textit{if} \qquad B \to 0. \tag{0.5}$$

(iii) *If S corresponds to a single Bloch energy band E_* for the periodic operator $H_{\Gamma,B=0}$, then $\mathcal{H}_{\mathrm{r}} \equiv \ell^2(\mathbb{Z}^2)$. Moreover if the dispersion law for E_* can be approximated as $E_*(k_1,k_2) = 2\cos(k_1) + 2\cos(k_2) + Bf(k_1,k_2)$, with k_1 and k_2 the Bloch momenta, then*

$$H^{\mathrm{eff}}_B = H^{(B)}_{\mathrm{Hof}} + \mathcal{O}(B) \qquad \textit{if} \qquad B \to 0. \tag{0.6}$$

Theorem 0.1 implies the following answers for Q.1) and Q.2): $\Pi_B H_{\Gamma,B} \Pi_B$ and $H^{(B)}_{\mathrm{Hof}}$ are unitarily equivalent up to an error which goes to zero if $B \to 0$ (*asymptotic unitary equivalence*); the dynamics generated by $H^{(B)}_{\mathrm{Hof}}$ approximates the dynamics generated by $\Pi_B H_{\Gamma,B} \Pi_B$ up to a small error over any macroscopic time-scale $t \in [0,T]$.

Question Q.3) suggests to combine SAPT-techniques with the randomness induced by V_ω. However, one of the main ingredients of SAPT is the separation in fast and slow degrees of freedom induced by the periodic structure of $H_{\Gamma,B=0}$. This

[1] This assumption can be relaxed by introducing the notion of *adiabatically decoupled* energy subspace, cf. [**8**] or [**3**].

separation (mathematically highlighted by means of a Bloch-Floquet transform) identifies the fast part of the dynamics with the one inside the fundamental cell of Γ. The slow part is related to the motion at the boundary of adjacent cells and is controlled by the slow variation of the Bloch momenta induced by the weak, but non-zero, magnetic field $B \ll 1$. In order to include V_ω in this schema, one needs to assume that the randomness perturbs the periodic structure on a scale larger that the typical length of the crystal and which becomes larger and larger when $B \to 0$. In other words SAPT-tecniques are compatible only with B-dependent random potentials of type

$$V_{\omega,B}(x,y) := w_\omega\big(B^{-1}x, B^{-1}x\big) \tag{0.7}$$

with w_ω suitable random variables. If order to overcome the quite unphysical restriction (0.7) one has to replace the usual Bloch-Floquet transform with some non-commutative extension. An hint in this direction is provided by the Bellissard's idea of replacing the Bloch-Floquet decomposition with the non-commutative notion of crossed product C^*-algebra [**2**].

Strong magnetic field limit. The opposite regime of a strong magnetic field $B \gg 1$ (accessible to experiments by means of optical lattices) is mathematically easier to treat. In this regime, the dominant terms for the "renormalized" Hamiltonian $B^{-1}\ H_{\Gamma,B}$ turns out to be a harmonic oscillator which fixes the energy threshold (Landau level). Under the assumption $V_\Gamma(x,y) \simeq 2\cos(x) + 2\cos(y)$, the first relevant term for the asymptotic description of the spectral properties of (0.1) is given by the *Harper* (effective) *model*

$$\big(H^{(B)}_{\mathrm{Har}}\xi\big)(s) := \xi\big(s + B^{-1}\big) + \xi(s - B^{-1}) + 2\cos(2\pi s)\xi(s), \qquad \xi \in L^2(\mathbb{R}). \tag{0.8}$$

The effective operator (0.8) was firstly derived by M. Wilkinson (1987). However, a rigorous (asymptotic) unitary derivation of $H^{(B)}_{\mathrm{Har}}$ from $H_{\Gamma,B}$, in the spirit of the Theorem 0.1, has been established only recently in [**4**]. Moreover, the proof can be successfully extended to the case of a random potential V_ω generalizing a perturbative technique developped in [**5**].

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